

H STORIA MATHEMATICA 12 (1985), 261–268

NOTE

Three Alternate Methods of Obtaining the Ancient Egyptian Formula for the Area of a Circle

PAULUS GERDES

*Department of Mathematics and Physics, Faculty of Education, Eduardo Mondlane University,
C.P. 257, Maputo, Mozambique*

INTRODUCTION

To determine the area of a circle, mathematicians in ancient Egypt used the square of $8/9$ of its diameter. This implies an approximation of π .

$$\pi \approx 4 \left(\frac{8}{9} \right)^2 = \frac{256}{81} \approx 3.1605,$$

which is indeed “a great accomplishment” [van der Waerden 1954, 32]. However, as K. Vogel has emphasized, “Just how this remarkably close approximation was found, we do not know” [Gillings 1972, 142]. After a short description of earlier attempts to explain how the Egyptian formula could have been found, we present three alternate methods.

EARLIER CONJECTURES

Vogel himself formulated an interesting conjecture concerning the origin of this Egyptian procedure. It uses a *semiregular octagon* that approximates the circle rather well (see Fig. 1). The area of the octagon $ABCDEFGH$ is equal to $7/9$ (or $63/81$) of the square of the diameter $[1]$, i.e., almost equal to the desired quantity of $(64/81)d^2$.

Using a variation on the theme of the semiregular octagon, R. Gillings [1972, 143–146] arrived at an alternate method of obtaining the formula. Once again, there is a jump from 63 to 64. But in the words of Gillings, the Egyptian scribe A’h-mosè “certainly knows his method is not *exact* . . . his method allows him to find a square nearly equal to a circle, so that we can, ‘*en caprice*,’ as it were, credit A’h-mosè with being the first authentic circle-squarer in history!” [1972, 145].

To H. Engels the conjecture of the semiregular octagon seems to be “*too complicated*” [1977, 137]. He suggested a simpler alternative and was the first to relate his hypothesis to material production: Egyptian stone masons covered their de-

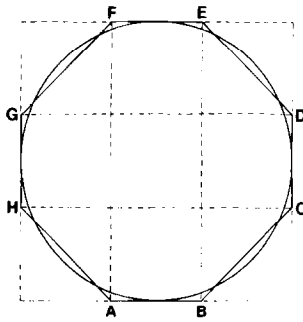


FIGURE 1

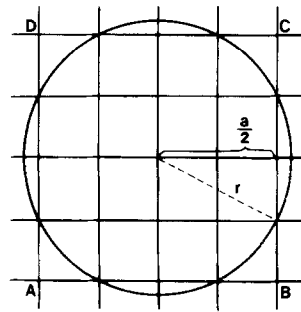


FIGURE 2

signs and walls in order to form a relief with orthogonal nets. Then the cutting points of the lattice lines and the contours of the design were carried over in fixed ratios. What can be said about circles that appear in these orthogonal nets?

The circle in Fig. 2 has, “intuitively,” the same area as the square $ABCD$. By dividing every square into 16 equal subsquares, the Egyptians could have discovered, Engels argues, that $a/2 = 8r/9$, or that $a = 8d/9$, where a denotes the length of the side of square $ABCD$. In fact, a is not exactly equal to $8d/9$, but, using the Pythagorean theorem, to $2d/\sqrt{5}$. However, the relative error is less than 0.62% [Engels 1977, 139].

CONJECTURES SUGGESTED BY AN EXAMINATION OF AFRICAN CRAFT TECHNIQUES

Two geometric designs related to the circle are widespread in Africa: the so-called *snake curve* (see Fig. 3) and the set of *equidistant concentric circles* (see Fig. 4). An examination of some examples of these circular patterns suggests several alternate conjectures for the origin of the Egyptian formula for the area of a circle.

1. A spiral-of-Archimedes-like figure represents symbolically the snake in Angola [Mveng 1980, 54]. These “snake curves” have been carved on wooden doors in Nigeria [Denyer 1978, 89] and may also be found in the fabrication of

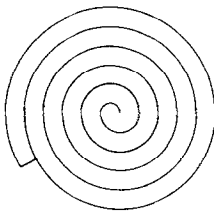


FIGURE 3

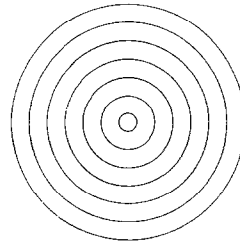


FIGURE 4

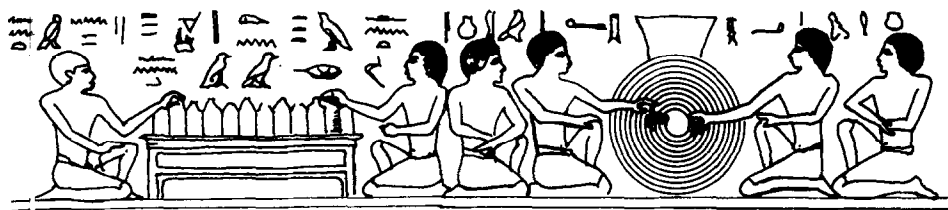


FIGURE 5

baskets (e.g., Sudan, modern Egypt) and mats (e.g., Mozambique). At the same time, this spiral pattern is a common motif for the wall decorations of burial sites in ancient Egypt (e.g., near Deir el-Medina). As early as ca. 2600 B.C. the “snake game” was played on a spiral-circular table [Erman & Ranke 1963, 327] (Fig. 5). Under Ramses III (1198–1167 B.C.), royal bread took a spiral form [Erman & Ranke 1963, 257] (Fig. 6). A description of the making of a spiral sisal mat will illustrate our first conjecture.

A string of sisal is rolled around a fixed point (see Fig. 7) and then sewn into successive spirals. Normally the end will be cut off a little bit in order to give the impression of a circle (see Fig. 8). Let us suppose that artisans in ancient Egypt knew how to make similar mats.

A string of sisal can be considered as a rectangle, whose width is taken as the unit of measurement. When the length L of the string is equal to a square n^2 , the area A of the string is equal to a square with side n . It may be assumed, of course, that rolling the string does not change its area. Now the *diameter* d of the “spiral circle” can be counted from the endpoint of the spiral to the opposite side (see Fig. 7). For $n = 8$, we get, *experimentally*, $d = 9$. In this manner we obtain

$$A = L = n^2 = \left(\frac{8}{9}d\right)^2,$$

i.e., exactly the ancient Egyptian formula for the area of a circle (Fig. 9).



FIGURE 6

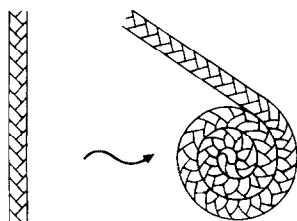


FIGURE 7

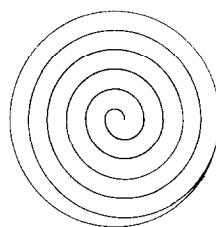


FIGURE 8

But why choose $n = 8$, and not, say, $n = 6$? The answer is found by investigating the length L of that part of the string necessary in order to get a "spiral-circle" with a natural number W of complete windings as a function of W . Table I gives the experimentally derived relationship between W and L ($d = 2W - 1$). The values of L have been rounded off. The smallest value of n for which L is a perfect square occurs when $W = 5$, i.e., $d = 9$ and $n = 8$. Hence the choice of $n = 8$. There remains the *practical* problem of *how* to count the complete windings: When is the first winding completed? Once $n = 8$ is chosen, it follows immediately, as we have seen, that $A = (8d/9)^2$.

In formulating this conjecture, we presupposed that artisans in ancient Egypt knew how to fabricate spiral circular mats more or less in the way described. The frequent interaction between ancient Egyptian and sub-Saharan cultures is an important but not a sufficient justification. What other evidence supports the assumption? Figure 10 shows a detail of a relief in the tomb of Ptakotep (ca. 2378 B.C.) in Sakkara. Hunters are catching birds, using a rope to close the clap-net. The end of the string is rolled up into a spiral. Unable to use perspective, the artist drew the spiral in a vertical position (cf. Fig. 5). Besides, a horizontally coiled rope seems rather unrealistic in a hunting scene. The presence of a "snake curve" in this relief as an artistic motif alone is unlikely. It may be concluded that rolling up strings into "spiral circles" was known in cultural contexts other than hunting. The fabrication of mats could well have been such a context. This familiarity with coiled strings was achieved at least 700 years before the scribe A'h-mosè (about 1650 B.C.) [Gillings 1972, 45] gave his famous formula for the area of a circle. Moreover, it is well known that stretched ropes were used to measure length in ancient Egypt [van der Waerden 1954, 15].

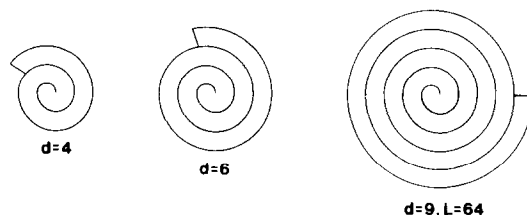


FIGURE 9

TABLE I

<i>W</i>	2	3	4	5	6
<i>L</i>	7	20	39	64	95
<i>d</i>	3	5	7	9	11

TABLE II

<i>T</i>	1	2	3	4	5
<i>L</i>	7	20	39	64	95
<i>d</i>	3	5	7	9	11

2. Yet another possibility is suggested by a set of equidistant concentric circles appearing as wall decorations (e.g., Tanzania [Denyer 1978, 41]), on tissues (e.g., Cameroun [Mveng 1980, 121]), and on roofs (e.g., Guinea [Denyer 1978, 143]). As a symbol for cosmic circles it is found in sculpture (e.g., Angola [Mveng 1980, 79]) and even in ancient rock paintings in Mozambique [Oliveira 1975, 21]. At the same time, this geometric pattern can also be encountered in ancient Egypt, e.g., as successive circular necklaces of Kagemni in Sakkara (ca. 2280 B.C.) or on the blue crown of Ramses II [Wiesner 1971, 168].

Circular sisal mats, in a pattern of equidistant concentric circles, are made in a manner similar to the method described for spiral mats. One cuts out a little circle, whose diameter equals the width of the string. This little circle serves as “center circle.” Around the center circle the mat is fabricated ring by ring (see Fig. 11), sewing together the successive rings. Each time, that part of the string that completely fills up the corresponding ring is cut off.

Let us suppose that Egyptian artisans made circular mats in the way described here. To obtain a mat having *T* rings, what will be the length of string *L* needed? As before, let us take the width of the string as the unit of measurement. Table II gives rounded-off experimental results. The smallest values of *T* and *d* (*d* = 2*T* + 1) for which *L* becomes equal to a square, *n*², are *T* = 4 and *d* = 9, respectively (see Fig. 12), with *n* = 8. In this case, we have *n* = (8/9)*d* and, as a consequence, we arrive once more at the ancient Egyptian formula for the area of a circle:

$$A = L = n^2 = \left(\frac{8}{9} d\right)^2.$$

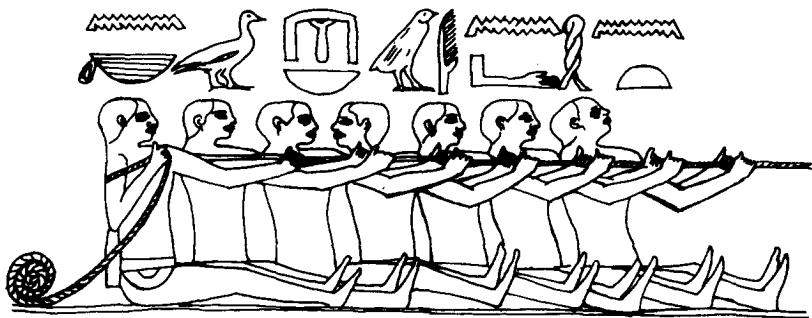


FIGURE 10

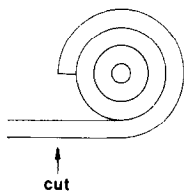


FIGURE 11

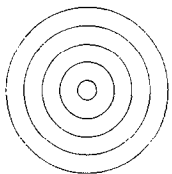


FIGURE 12

CONJECTURE SUGGESTED BY PLAYING WITH CIRCULAR OBJECTS

A board game, frequently referred to by its Arabic name *mancala* [Zaslavsky 1972, 118], is very popular throughout Africa. In ancient Egypt it was played on a board with three rows of fourteen holes and a storage pot [Zaslavsky 1972, 126]. The counters of this game are almost-spherical seeds, pebbles, or beans. The snake game was also played with little balls. As Fig. 5 shows, Egyptian draughtsmen were small upright cylinders. These and other games were not the only cultural settings where one encounters many equal circular, cylindrical, or spherical objects. Weighing rings and making beads [Erman & Ranke 1963, 623, 643] also represented such situations in ancient Egypt.

In contexts like these, e.g., when a participant in a game waits for the moves of another participant, it is quite natural that one starts to play with the stones, beans, or rings themselves. One may well “draw,” transform, and count geometric patterns (see the examples in Fig. 13). Now one may ask—and an Egyptian player, artisan, or scribe could well have asked—is it possible to “draw” greater circles using the little circles, i.e., with the small spherical balls or cylindrical draughtsmen? A simple way to obtain greater circles is to build them up ring by

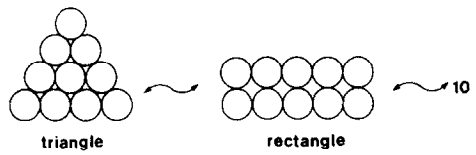


FIGURE 13

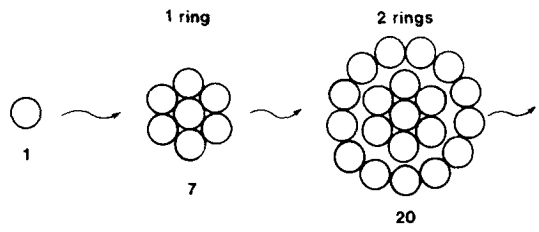


FIGURE 14

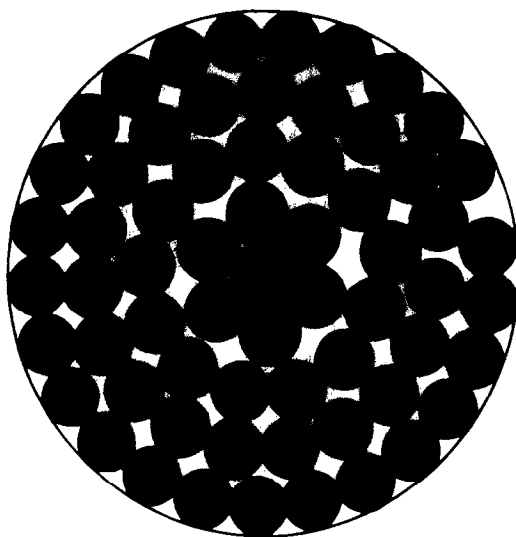


FIGURE 15

ring, as Fig. 14 shows. Let d denote the diameter of the great circle and D the diameter of a little circle. For convenience, we may take $D = 1$. By *experiment* it is found that $1 + 6 + 13 + 19 + 25$, or 64, little circles go into a great circle of diameter 9 (see Fig. 15). It is the smallest diameter for which T , the total number of little circles, is a perfect square. The same 64 little circles also “fill up” a square of side 8 (see Fig. 16). In this manner, an Egyptian may well have concluded that a great circle of diameter 9 and a square of side 8 have—more or less—the same area, because both are “covered” by the same number (64) of little circles.

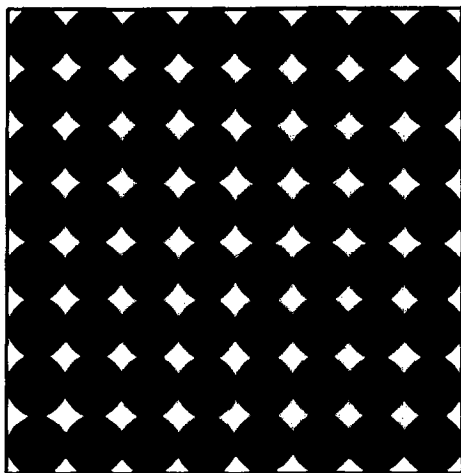


FIGURE 16

ACKNOWLEDGMENTS

The author is grateful to Professor H. Engels (Aachen, FRG), Dr. G. Baumbach (Jena, GDR) and Dr. W. Willemer (Dresden, GDR) for their valuable comments on this paper, and to Atilio Armando for his drawings.

NOTE

1. Following Vogel's construction until $7/9$, we have a rather didactical method for deriving a useful value for π in primary schools [Muravarava 1983]: $\pi \approx 4 \cdot (7/9) \approx 3.11$.

REFERENCES

- Denyer, S. 1978. *African traditional architecture*. London: Heinemann.
- Engels, H. 1977. Quadrature of the circle in Ancient Egypt. *Historia Mathematica* **4**, 137–140.
- Erman, A., & Ranke, H. 1963. *La civilisation Égyptienne*. Paris: Payot.
- Gillings, R. 1972. *Mathematics in the time of the pharaohs*. Cambridge, Mass.: MIT Press.
- Martin, H. 1929. *L'art égyptien*. Paris: Flammarion.
- Muravarava, P. 1983. Os Egípcios Antigos e a área do círculo. *Tlanu* **2**, 34–45 (Maputo, Mozambique: Faculty of Education).
- Mveng, R. 1980. *L'art et l'artisanat africains*. Yaoundé, Cameroun: Éditions Cle.
- Oliveira, O. de. 1975. *Gravuras e pinturas rupestres de Moçambique*. Beira, Mozambique: Empresa Moderna.
- Waerden, B. van der 1954. *Science awakening*. Groningen: Wolters.
- Wiesner, J. 1971. *Egipto*. Lisbon: Editorial Verbo.
- Zaslavsky, C. 1973. *Africa counts, number and pattern in African culture*. Boston: Prindle, Weber & Schmidt.